Abstract—This paper introduces the use of projective invariants for object recognition using omnidirectional vision. The catadioptric image formation is modeled with a two-step mapping via the sphere using the conformal geometric algebra. Furthermore, we show that the projective invariants in a plane are equivalent to the invariants of circles in the unit sphere. This equivalence induces us to project features from the catadioptric image to the sphere, calculate their invariants and use these invariants to recognize objects.

Index Terms—Omnidirectional Vision, Catadioptric, Invariants, Conformal Geometric Algebra.

I. INTRODUCTION

This paper shows that the geometric algebra can be used to calculate the projective invariants, which are based in the cross ratio. We use these invariants to solve the object recognition task using an omnidirectional vision system. The projective invariants do not hold for the catadioptric image, so we must do something to get them back again. In this work we will explain how that can be done, using the conformal geometric algebra.

The rest of this paper is organized as follows: We give a brief description of the geometric algebra and also of the conformal geometric algebra in section II. In section III we explain the projective invariants. In section IV we give a brief introduction to the omnidirectional vision in terms of the conformal geometric algebra. In section V we explain the projective invariants and the omnidirectional vision. An application is given in section VI, and finally the conclusions are in section VII.

II. GEOMETRIC ALGEBRA

In the simplest sense, we can see the geometric algebra as a set of rules acting on objects that allow us to treat geometric things with the tools we learned to use in algebraic tasks. In other words, with geometric algebra we have a standard framework to deal with numbers and its operations (algebra) but also with magnitudes like line segments, areas, volumes and angles (geometry). With the union of geometry and algebra we have a powerful framework to deal with physical problems.

In general, a geometric algebra \( G_n \) is a \( n \)-dimensional vector space \( \mathcal{V}^n \) over the reals \( \mathcal{R} \). We also denote with \( G_{p,q,r} \) a geometric algebra over \( \mathcal{V}^{p,q,r} \) where \( p, q, r \) denote the signature \( p, q, r \) of the algebra. If \( p \neq 0 \) and \( q = r = 0 \) the metric is euclidean \( G_n \), if just \( r = 0 \) the metric is pseudoeuclidean \( G_{p,q} \) and if non of them are zero the metric is degenerate.

We will use the letter \( e \) to denote the vector basis \( e_i \). In a geometric algebra \( G_{p,q,r} \), the geometric product of two basis vectors is defined as

\[
e_i e_j = \begin{cases} 
1 & \text{for } i = j \in 1, \ldots, p \\
-1 & \text{for } i = j \in p + 1, \ldots, p + q \\
0 & \text{for } i = j \in p + q + 1, \ldots, p + q + r \\
e_i \wedge e_j & \text{for } i \neq j \end{cases}
\]

A. Subspaces, Blades and Simplexes

In geometric algebra we can represent different subspaces with different geometric interpretations using blades. In other words for every \( r \)-dimensional subspace we have an \( r \)-blade \( A_r \) such that the subspace is the solution set of the equation

\[x \wedge A_r = 0, \text{ for } x \in \mathcal{V}^n\]

As we can see in the last equation, \( A_r \) is unique to within a nonzero scalar factor. The blade \( A_r \) can be interpreted as direct measure (or \( r \)-volume) on the subspace, with orientation and magnitude \( |A_r| \). Note that \( A_r \) determines a unique subspace and \(-A_r\) determines the same subspace of vector but with the opposite orientation.

An \( r \)-simplex (\( r \)-dimensional simplex) in \( \mathcal{V}^n \) is the convex hull of a \( r + 1 \) set of points of which at least \( r \) are linearly independent. The frame of the simplex is the set \( \{a_0, a_1, a_2, \ldots, a_r\} \) of points that define the simplex. One of the points, say \( a_0 \) is distinguished and called the base point of the simplex.

\[A_r \equiv a_0 \wedge a_1 \wedge a_2 \cdots \wedge a_r = a_0 \wedge \mathcal{F}_r\]

\[\mathcal{F}_r \equiv (a_1 - a_0) \wedge (a_2 - a_0) \wedge \cdots \wedge (a_r - a_0),\]

\[\mathcal{F}_i \equiv a_i - a_0 \text{ for } i = 1 \ldots r.
\]
We call \( \overline{A_r} \) the tangent of the simplex because it is tangent for the \( r \)-plane in which it lies. This tangent assigns a definite orientation to the simplex, but note that if the simplex passes through the origin its moment vanishes. To avoid this problem we embed \( \mathbb{V}^n \) as a \( n \)-plane in a higher dimension space. Thus all points are treated equal and the moment never vanishes.

**B. Conformal Geometric Algebra**

In the Euclidean space the composite of displacements is complicated because rotations are multiplicative but translations are additive. In order to make translations multiplicative too, we use the Conformal Geometric Algebra.

In the generalized homogeneous coordinates for points in the euclidean space, we need that they be null vectors and also lie on the intersection of the null cone \( \mathcal{N}^{n+1} \) (the set of all null vectors) with the hyperplane

\[
\mathcal{P}^{n+1}(e, e_0) = \{ X \in \mathbb{R}^{n+1,1} | e(X - e_0) = 0 \},
\]

that is

\[
\mathcal{N}^n_e = \mathcal{N}^{n+1} \cap \mathcal{P}^{n+1}(e, e_0) = \{ x \in \mathbb{R}^{n+1,1} | X^2 = 0, X \cdot e = -1 \}
\]

which is called the homogeneous model of \( \mathcal{E}^n \), also called the horosphere (see Fig. 2) in hyperbolic geometry.

The points that satisfy the restrictions \( X^2 = 0 \) and \( X \cdot e = -1 \) are

\[
X = x + \frac{1}{2}x^2 + e_0
\]

where \( x \in \mathbb{R}^n \) and \( X \in \mathcal{N}^n \). Note that this is a bijective mapping. From now and in the rest of the paper the conformal points will be denoted by an italic uppercase letter (\( X \)), and the Euclidean points will be denoted by bold lowercase letters \( x \).

In table I we show the geometric entities of the conformal geometric algebra. Note that in the IPNS representation the point is a sphere with radius zero. In the dual representation the sphere is calculated using 4 points that lie on it.

**TABLE I**

<table>
<thead>
<tr>
<th>Entity</th>
<th>IPNS Representation</th>
<th>OPNS (Dual) Representation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sphere</td>
<td>( x = \frac{1}{2}(p^2 - p^2 + e) + e_0 )</td>
<td>( X = A \backslash B \cup D )</td>
</tr>
<tr>
<td>Point</td>
<td>( X = x + \frac{1}{2}x^2 + e_0 )</td>
<td>( X^* = S_1 \cup S_2 \cup S_3 )</td>
</tr>
<tr>
<td>Plane</td>
<td>( n = (a-b) \wedge (a-c) )</td>
<td>( I^2 = A \wedge B \wedge C )</td>
</tr>
<tr>
<td>Line</td>
<td>( L^* = A \wedge B \wedge C )</td>
<td>( I^2 = A \wedge B \wedge C )</td>
</tr>
<tr>
<td>Circle</td>
<td>( Z = S_1 \cup S_2 )</td>
<td>( Z^* = A \wedge B \wedge C )</td>
</tr>
<tr>
<td>Point Pair</td>
<td>( PP = S_1 \cup S_2 \cup S_3 )</td>
<td>( PP^* = A \wedge B \wedge C )</td>
</tr>
</tbody>
</table>

1) **Simplexes and Conformal Points:** Evaluating the outer product of \( r \) linearly independent conformal points \( a_0, a_1, \ldots, a_r \), where \( r \leq n \) and \( n \) is the maximum grade of the algebra. The outer product of \( r \) conformal points is

\[
a_0 \wedge a_1 \wedge \cdots \wedge a_r = A_r + e_0A_r + \frac{1}{2}eA_r - \frac{1}{2}EA_r, \tag{7}
\]

where

\[
A_r = a_0 \wedge a_1 \wedge \cdots \wedge a_r,
\]

\[
A_r^+ = \sum_{i=0}^{r} (-1)^i a_0 \wedge \cdots \wedge \hat{a}_i \wedge \cdots \wedge a_r,
\]

\[
A_r^- = \sum_{i=0}^{r} (-1)^i a_0^2 \wedge \cdots \wedge \hat{a}_i \wedge \cdots \wedge a_r,
\]

\[
A_r^\pm = \sum_{i=0}^{r} \sum_{j=i+1}^{r} (-1)^{i+j} a_0 \wedge \cdots \wedge \hat{a}_i \wedge \cdots \wedge \hat{a}_j \wedge \cdots \wedge a_r \tag{8}
\]

Note that \( A_r \) is the moment of the simplex with tangent (boundary) \( A_r^+ \), which is in concordance with equation 3.

The outer product \( a_0 \wedge a_1 \wedge \cdots \wedge a_r \) represents a sphere when \( A_r = 0 \)

\[
a_0 \wedge a_1 \wedge \cdots \wedge a_r = -[e_0 - \frac{1}{2}cA_r^+(A_r^+)^{-1} + \frac{1}{2}A_r^+(A_r^+)^{-1}]EA_r^+ \tag{9}
\]
where the center and radius of the sphere

\[ c = \frac{1}{2} A^+_r (A^+_r)^{-1}, \]
\[ \rho^2 = c^2 + A^-_r (A^+_r)^{-1}. \]  

(10)

### III. Invariants

An invariant is a property that remains unchanged under certain class of transformation. Within the context of vision, we are interested in determining the invariants of an object under perspective projection onto an image. The cross-ratio of four collinear points is a well known 1D-invariant under projective transformations but it can be extended to 2D, so we can use it for image invariants. In the 2D case we need five points. In the 3D case we need six points. In the 3D space these invariants can be interpreted as the cross-ratio of tetrahedral volumes.

In order to explain the invariants using conformal points we will explain it using projective points first. An euclidean point \( x \) is represented as projective point with \( X = x + e_- \).

(11)

For the 1D case we need four points \( X_i \in G_{1,1} \); an example of a 1D invariant is

\[ Inv_1 = \frac{(X_3 \wedge X_4)I^{-1}_{p1}(X_4 \wedge X_2)I^{-1}_{p1}}{(X_4 \wedge X_1)I^{-1}_{p1}(X_3 \wedge X_2)I^{-1}_{p1}}, \]

where \( I_{p1} = e_1 \wedge e_- \) denotes the pseudoscalar of the 1D projective space.

If we use conformal points the outer product of two points leads to a point pair, so with four point pair we can compute the 1D invariants. Also note that we use the \( A_r \) (9) part of the point pair (the moment of the simplex) to calculate the invariant.

\[ P_1 = X_3 \wedge X_1 \]
\[ P_2 = X_4 \wedge X_2 \]
\[ P_3 = X_4 \wedge X_1 \]
\[ P_4 = X_3 \wedge X_2 \]  

(13)

Let \( A_{r,k} \) denote the \( A_r \) part of the k-point-pair \( P_k \) where \( k = 1 \ldots 4 \). Then the invariant using the moment \( A_r \) of the simplex is

\[ Inv_1 = \frac{A_{r,1}I^{-1}_{E} A_{r,2}I^{-1}_{E}}{A_{r,3}I^{-1}_{E} A_{r,4}I^{-1}_{E}}. \]  

(14)

Now, for the 2D case we need five points, an example of a 2D invariant is

\[ Inv_2 = \frac{(X_5 \wedge X_4 \wedge X_3)I^{-1}_{p2}(X_5 \wedge X_2 \wedge X_1)I^{-1}_{p2}}{(X_5 \wedge X_1 \wedge X_3)I^{-1}_{p2}(X_5 \wedge X_2 \wedge X_4)I^{-1}_{p2}}, \]

where \( I_{p2} = e_1 \wedge e_2 \wedge e_- \) denotes the pseudoscalar of the 2D projective space.

If we use conformal points the outer product of three points leads to a circle, so with four circles we can compute the 2D invariants. Also note that we use the \( A_r \) (9) part of the circle (the moment of the simplex) to calculate the invariant.

\[ C_1 = X_5 \wedge X_4 \wedge X_3 \]
\[ C_2 = X_5 \wedge X_2 \wedge X_1 \]
\[ C_3 = X_5 \wedge X_1 \wedge X_3 \]
\[ C_4 = X_5 \wedge X_2 \wedge X_4 \]  

(16)

Let \( A_{r,k} \) denote the \( A_r \) part of the k-circle \( C_k \) where \( k = 1 \ldots 4 \). Then the invariant using the moment \( A_r \) of the simplex is

\[ Inv_2 = \frac{A_{r,1}I^{-1}_{E} A_{r,2}I^{-1}_{E}}{A_{r,3}I^{-1}_{E} A_{r,4}I^{-1}_{E}}. \]  

(17)

Finally, for the 3D case we need six points, an example of a 3D invariant is

\[ Inv_3 = \frac{(X_1 \wedge X_2 \wedge X_4 \wedge X_5)I^{-1}_{p3}(X_1 \wedge X_3 \wedge X_4 \wedge X_5)I^{-1}_{p3}}{(X_1 \wedge X_2 \wedge X_4 \wedge X_6)I^{-1}_{p3}(X_1 \wedge X_3 \wedge X_4 \wedge X_5)I^{-1}_{p3}}, \]

where \( I_{p3} = e_1 \wedge e_2 \wedge e_3 \wedge e_- \).

The outer product of four conformal points leads to a sphere, so with four spheres we can compute the 3D invariants. Also note that we will use the \( A^+_r \) (9) part of the sphere (the tangent of the simplex) to calculate the invariant.

\[ S_1 = X_1 \wedge X_2 \wedge X_4 \wedge X_5 \]
\[ S_2 = X_1 \wedge X_3 \wedge X_4 \wedge X_5 \]
\[ S_3 = X_1 \wedge X_2 \wedge X_4 \wedge X_6 \]
\[ S_4 = X_1 \wedge X_3 \wedge X_4 \wedge X_5 \]  

(19)

Let \( A^+_{r,k} \) denote the \( A^+_r \) part of the k-sphere \( S_k \) where \( k = 1 \ldots 4 \). Then the invariant using the tangent \( A^+_r \) of the simplex \( A_r \) is

\[ Inv_3 = \frac{A^+_{r,1}I^{-1}_{E} A^+_{r,2}I^{-1}_{E}}{A^+_{r,3}I^{-1}_{E} A^+_{r,4}I^{-1}_{E}}. \]  

(20)

Note that in this case we use \( A^+_r \) instead of \( A_r \) because when we make the outer product of \( r \) conformal points and \( r \) is equal to the grade of the algebra the \( A_r \) part is zero.
IV. OMNIDIRECTIONAL VISION

We known, that traditional perspective cameras have a narrow field of view. One effective way to increase the visual field is the use of a catadioptric sensor which consists of a conventional camera and a convex mirror. In order to be able to model the catadioptric sensor geometrically, it must satisfy the restriction that all the measurements of light intensity pass through only one point in the space (effective viewpoint). The complete class of mirrors that satisfy this restriction where analyzed by Baker and Nayar [1].

It has been shown that we can model any catadioptric projection through a spherical projection. We just need a point that passes through the optical axis at a distance \( l \) to the focus and a plane that is perpendicular to the optical axis and is at a distance \( m \) of the focus. The parameters \( l \) and \( m \) are calculated according to the eccentricity and scaling parameters of the mirror. The projections are equivalent if we choose

\[
l = \frac{2\epsilon}{1 + \epsilon^2} \quad \text{and} \quad m = \frac{\mu - \epsilon(\epsilon\mu + 2\lambda - 2)}{1 + \epsilon^2}.
\]

A. Omnidirectional Vision and Geometric Algebra

The catadioptric image formation can be modeled using the conformal geometric algebra. First, we assume that the optical axis of the mirror is parallel to the \( e_2 \) axis, then let \( f \) be a point in the Euclidean space (which represents the focus of the mirror lying in such optical axis) with conformal representation given by

\[
F = f + \frac{1}{2} f^2 e + e_0.
\]

Using the point \( F \) as the center, we define the unit sphere \( S \) (see Fig. 3) as follows

\[
S = F - \frac{1}{2} e.
\]

Now let \( N \) be the point of projection (that also lies on the optical axis) at a distance \( l \) of the point \( F \), this point can be found using a translator

\[
T = 1 + \frac{l e_2 e}{2},
\]

and then

\[
N = TF^T.
\]

Finally, the image plane \( \Pi \) is perpendicular to the optical axis at a distance \(-m\) from the point \( F \) and its equation is

\[
\Pi = e_2 + (f \cdot e_2 - m)e.
\]

1) Point projection: Let \( p \) be a point in the Euclidean space, the corresponding conformal point in the conformal space is

\[
P = p + \frac{1}{2} p^2 e + e_0.
\]

Now, for the projection of the point \( P \) we trace a line joining the points \( F \) and \( P \). Using the definition of the line in dual form we get

\[
L_1^* = S \wedge P \wedge e.
\]

Note that the outer product of a sphere (in its normal form (IPNS)), a point and the point at infinity produces a line that passes through the point and center of the sphere. Then, we calculate the intersections of the line \( L_1^* \) and the sphere \( S \) which result in the point pair

\[
P_1^* = L_1^* \cdot S.
\]

From the point pair we choose the point \( P_1 \) which is the closest point to \( P \)

\[
P_1 = \frac{P_1^* + \sqrt{P_1^* \cdot P_1^*}}{P_1^* \cdot e}
\]

and then we find the line passing through the points \( P_1 \) and \( N \)

\[
L_2^* = P_1 \wedge N \wedge e
\]

Finally we find the intersection of the line \( L_2^* \) with the plane \( \Pi \) with

\[
Q = L_2^* \cdot \Pi.
\]

The point \( Q \) is the projection in the image plane of the point \( P \) of the space. Notice that we can project any point in the space into any type of mirror (changing \( l \) and \( m \)) using the previous procedure (see Fig. 3).

Fig. 3. Conformal Unify Model and Point Projection.
B. Inverse point projection

We have already seen how to project a point in the space to the image plane through the sphere. But now we want to back-project a point in the image plane to the 3D space. First, let Q be a point in such image plane; the equation of the line passing through the points Q and N is

\[ L_2 = Q \wedge N \wedge e, \]

and the intersection of the line \( L_2 \) and the sphere \( S \) is

\[ P_P = L_2^* \cdot S. \]

From the point pair we choose the point \( P_1 \) which is the closest point to Q

\[ P_1 = \frac{P_P + \sqrt{P_P \cdot P_P}}{P_P \cdot e}, \]

then we find the equation of the line passing through point \( P_1 \) and the focus \( F \)

\[ L_1^* = P_1 \wedge F \wedge e. \]

The point \( P \) lies on the line \( L_1^* \), but it can not be calculated exactly because a coordinate has been lost when the point was projected to the image plane (a single view does not allow to know the projective depth). However, we can project this point to some plane and say that it is equivalent to the original point up to scale factor.

V. INVARIANTS AND OMNIDIRECTIONAL VISION

In section III we have seen how to calculate projective invariants from circles using the conformal geometric algebra. In section IV we also have seen how to model the omnidirectional vision system using Conformal Geometric Algebra. Now, we will see how to use both, the omnidirectional vision and the projective invariants.

The projective invariants do not hold in the catadioptric image, but they do in the image sphere. Therefore we must take some points in the catadioptric image and project them to the sphere. Once we do this we can proceed to calculate the invariants using four circles.

First we will show briefly that projective invariants in the plane are equivalent to projective invariants in the \( S^2 \) sphere (image sphere), see Fig. 4. According to section IV we define the point \( F \) (in this case it will be equal to \( e_0 \)), then the unit sphere is

\[ S = e_0 - \frac{1}{2} e, \]

the point \( N \) and the plane \( \Pi \) are defined in (25) and (26) respectively.

Now, let \( x_1, x_2, ..., x_5 \) be points in the euclidean space with conformal representation

\[ X_i = x_i + \frac{1}{2} x_i^2 e + e_0, \quad \text{for } i = 1 \ldots 5. \]

Then using the equations 28 to 30 we project the points in the space to the sphere and that give us the projected points say \( U_1, U_2, \ldots U_5 \).

In the other hand, the image plane \( \Pi_f \) (in order to compare the invariants) is defined as

\[ \Pi_f = e_2 + e. \]

To project the points to the plane we first use (28), then we intersect the plane with each line

\[ Q_i = L_1^* \cdot \Pi_f \text{ for } i = 1 \ldots 5. \]

The point \( Q_i \) is a flatpoint which is the outer product of a conformal point with the null vector \( e \) (the point at infinity). To obtain the conformal point from the flatpoint we can use

\[ V_i = \frac{Q_i \wedge e_0}{(-Q_i \cdot E)E} + \frac{1}{2} \left( \frac{Q_i \wedge e_0}{(-Q_i \cdot E)E} \right)^2 e + e_0. \]

Using (16) we calculate the two sets of four circles, one for the points \( U_i \) and one for \( V_i \). With each set of circles we calculate the two invariants using (17), after comparing this two invariants we will see that them are the same. Therefore, we now know that if we project the points in the catadioptric image to the sphere we have again the projective invariants.

![Fig. 4. Different views of points in the space projected to the (image) sphere and to the (image) plane used to compare the calculated invariants. a) Global view of points projected to the sphere and to the plane, b) Points projected in the sphere with the circles formed to calculate the invariants and c) Points projected in the plane with its circles formed to calculate the invariants.](image-url)
VI. Experiments

The hardware setup used to implement the object recognition and grasping is basically a mobile robot equipped with an omnidirectional vision system (which consists in a parabolic mirror and a video camera), a stereo vision system mounted in a pan-tilt unit and a robot arm. The robot has a Pentium III processor running Linux, it also has a wireless LAN adapter so we have a cable free connection to it using a desktop PC (see Fig. 5).

The omnidirectional image has the advantage of a bigger field of view. This capability allows to see all the objects around the robot without moving it. In contrast to the stereo system, which does not see all the objects or in some cases none of them (see Fig. 7).

![Fig. 5. Mobile robot and omnidirectional vision system](image)

Before we use the omnidirectional system we must calibrate it with this we mean find the mirror center, focal length, skew and aspect ratio. A good reference for paracatadioptric calibration is [5].

Once that we have calibrated the system we can continue with the experiment. First we put three tables around the robot each one of them having several objects, the objective is that the robot recognize an object given by the user from the different objects in the tables.

The recognition process consists of various steps that are shown in Fig. 6, the recognition system is similar to previous systems [14] in all but the projection of the features to the sphere stage.

To recognize an object we first take features from the catadioptric image, then these features are projected onto the unit sphere using Eqs. (33 - 35). With this features in the sphere we calculate the circles formed with them (see Eq. 16). Finally, the invariants are calculated with Eq. 17 which are equivalent to the projective invariants. These invariants are compared with the previously acquired invariants in the library to identify the object.

The key points of an object are selected by hand. If they are accurate enough, our procedure can recognize the objects correctly. In general this kind of invariants are a bit sensitive to noise, due to the computations. We strongly believe that projective invariants are a good approach for object recognition and manipulation. In order to diminish the effect of noise in the data, we can compute several invariants related with the object, so that the accuracy of the recognition is increased. Utilizing an automatic corner detector the procedure of object recognition using our method can be carried out in real time.

Once that the object is recognized we rotate the robot until the object is in front of the stereo system. Since the object is now visible to the stereo camera, we can use an inverse kinematic approach to grasp the object. In our case we chose for the approach of [12] which is very interesting. Such approach models the joints of the robot arm using spheres, circles, lines and planes which are entities very easy to handle in conformal geometric algebra. In Fig. 8 we show the robot grasping an object.

![Fig. 6. The recognition system.](image)

![Fig. 7. Initial state of the experiment, a) Omnidirectional view, b) Left image of the stereo system and c) Right image of the stereo system.](image)
VII. CONCLUSION

In this paper, we use the conformal geometric algebra as a framework for the catadioptric image formation. This framework also allowed us to calculate the invariants of circles in the sphere, which are equivalent to the invariants in a plane. These two ideas are related applying the inverse projection of the catadioptric image formation to the features in the catadioptric image. Finally, the calculated invariants are used to recognize objects with the advantage of the bigger field of view offered by the omnidirectional vision system.

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